$\boxed{\text{Bellman functions}}$

Let's consider a process solution of

$$Y_0 = y_0$$

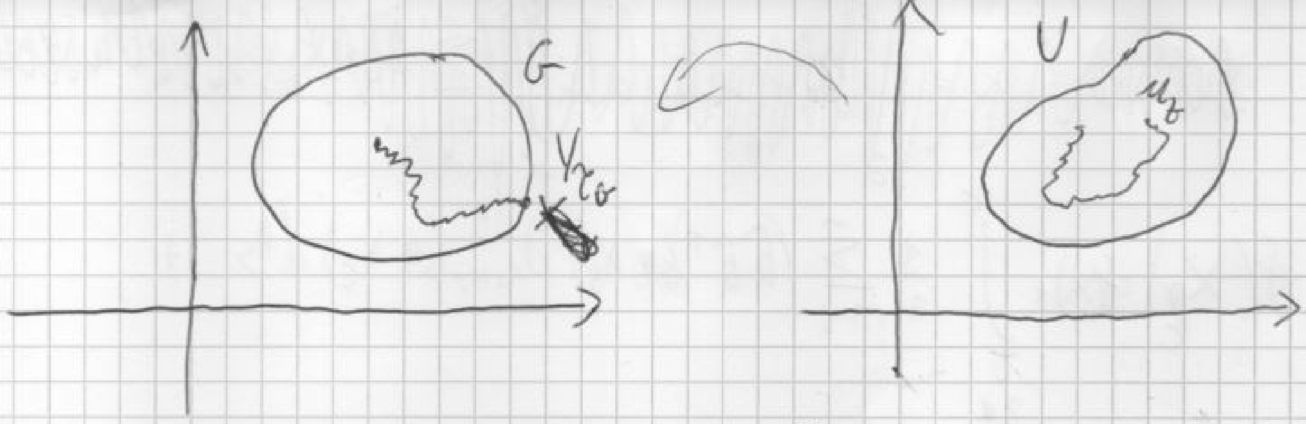
$$dX_t = b(X_t, u_t)\,dt + \sigma(X_t, u_t)\,dB_t \; ;$$

Where $(t,\omega) \longmapsto u_t(\omega)$ is a control process such that $u_t(\omega) \in U \subseteq \mathbb{R}^m$ where U is a given borel set

Let's consider a borel set $G \subseteq \mathbb{R}^n$

Let's consider a payoff density F and a bequest function K

$$F : \mathbb{R}^n \times U \longrightarrow \mathbb{R} \; ; \; K : \mathbb{R}^n \longrightarrow \mathbb{R}$$



$$\tau_v(\omega) = \inf\{t \geq 0 \mid X_t(\omega) \notin G\}$$

We define the average performance function $J^u(x)$ for $x \in G$ by

$$J^u(y) = E^y\left[\int_0^{\tau_v} F(X_s, u_s)\,ds + K(X_{\tau_v})\,\chi_{\{\tau_v < +\infty\}}\right]$$

$\boxed{\text{Def}}$ (Bellman function)

$$B(x) = \sup_{\substack{u=\{u_t\}_{t\geq 0} \\ u \text{ admissible}}} J^u(y) \quad \text{over the set of admissible controls}$$

Markov controls $\longleftrightarrow$ $u(t,\omega) = u_0(t, Y_t(\omega))$

if $u^* = \{u^*_t\}_{t\geq 0}$ is such that $B(y) = J^{u^*}(y)$ $\forall x$

then u^* is called an optimal control

$\boxed{\text{Teor}}$ $\boxed{\text{HJB equation}}$ Let $L^v f(x) = \sum_{i=1}^{m} b_i(x,v)\frac{\partial f}{\partial x_i}(x)$

$$+\frac{1}{2}\sum_{i,j=1}^{m}(\sigma\sigma^T)_{i,j}(x,v)\frac{\partial^2 f}{\partial x_i \partial x_j}(x)$$

$\boxed{\text{Obs}}$ L^v is the infinitesimal generator of the process

$$X_t = X_0 + \int_0^t b(X_s, v)\,ds + \int_0^t \sigma(X_s, v)\,dB_s$$

Let's consider the bellaman function

$$B(y)=\sup\{J^u(y) \mid u=u_0(Y) \text{ Markov Control}\}$$

Let's assume that B satisfies

$$E^y\left[|B(Y_\alpha)|+\int_0^\alpha |L^v B(Y_t)|\,dt\right]<+\infty$$

for all bounded stopping times $\alpha<T$, where $T^{(\omega)}=\inf\{s>0 \mid Y_s(\omega)\notin G\}$
$\tau_G(\omega)$

$\forall y\in G$; $\forall v\in U$. Suppose that T is $T<+\infty$ almost surely w.r.t. Q^y for all $y\in G$. Suppose that an optimal markov control $u^*=u_0^*(Y)$ exists.

Let's suppose that ∂G is regular for Y^{u^*}.

THEN. (1) $\sup_{v\in U}\{F^v(y)+(L^v B)(y)\}=0 \quad \forall y\in G$

and

(2) $B(y)=K(y) \quad \forall y\in\partial G$

Moreover, the supremum in (1) is obtained if $v=u_0^*(y)$ where $u^*=u_0^*(Y)$ is an optimal markov control. i.e.

$$B(y)=J^{u^*}(y)=E^y\left[\int_0^T F(Y_s,u^*(Y_s))\,ds+K(Y_T)\right]$$

$\boxed{\text{Def}}$ $x\in\partial G$ is regular for $X_t=b\,dt+\sigma\,dB_t$

if, given $\tau_G=\inf\{s>0 \mid X_s^x\notin G\}$,

we have $Q^x[\tau_G=0]=1$. ∂G is regular if x is regular $\forall x\in\partial G$

probability 1 starting from x starting point

Proof if $y \in \partial G$ is regular then $\tau_G = 0$ almost surely w.r.t. Q^y, so

$$B(y) = \sup_{u \text{ control}} E^y\left[\int_0^{\tau_G} F(Y_s, u_s)\,ds + K(Y_{\tau_G})\right]$$

but $\tau_G = 0$ a.s., so

$$B(y) = \sup_u E^y\left[0 + K(Y_0)\right]$$

but $Y_0 = y$ a.s., $\forall u$, so

$B(y) = K(y)$. OK. We are using the following theorem to prove our result

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Theor Under the same Hypotheses as the previous theorem, if we consider

$$w(y) = E^y\left[K(Y_{\tau_G})\right] + E^y\left[\int_0^{\tau_G} F^u(Y_s)\,ds\right]$$

for $x \in G$ then

a) $$\begin{cases} Aw = -F^u \text{ in } G & (3) \quad \text{almost surely w.r.t. } Q^y \\ \lim_{t \uparrow \tau_G} w(Y_t) = K(Y_{\tau_{\partial G}}) & (4) \quad \forall y \in G. \end{cases}$$

i.e. w is the solution of the Dirichlet Poisson problem

NB A is the infinitesimal generator of Y

b) If there exists a function $w_1 \in C^2(G)$ and $C > 0$ s.t.

$$|w_1(y)| < C\left(1 + E^y\left[\int_0^{\tau_G} |F^u(Y_s)|\,ds\right]\right) \quad \forall y \in D$$

and W_1 satisfies (1) and (2) then $W_1 = W$

So by the theorem we get, for $u = u^*$:

$$\underline{(L^{u^*} B)(y) = -F(y, u_0^*(y))} \quad \forall y \in G$$

because $B = W$ if $u = u^*$ OK.

Fix $y \in G$; let $u = u_0(Y)$ be a markov control.

Let $\alpha \leq T$ be a stopping time $T = \tau_G$

since $J^u(y) = E^y\left[\int_0^T F^u(Y_s)\,ds + K(Y_T)\right]$

We get

$$E^y\left[J^u(Y_\alpha)\right] = E^y\left[E^{Y_\alpha}\left[\int_0^T F^u(Y_s)\,ds + K(Y_T)\right]\right] =$$

$$\boxed{\text{Markov property}} \longrightarrow = E^y\left[E^y\left[\theta_\alpha\left(\int_0^T F^u(Y_s)\,ds + K(Y_T)\right) \Big| \mathcal{F}_\alpha\right]\right]$$

$$\boxed{\text{by a proper lemma}} = E^y\left[E^y\left[\int_\alpha^T F^u(Y_s)\,ds + K(Y_T) \Big| \mathcal{F}_\alpha\right]\right]$$

$$= E^y\left[\int_0^T F^u(Y_s)\,ds + K(Y_T) - \int_0^\alpha F^u(Y_s)\,ds\right] =$$

$$= J^u(y) - E\left[\int_0^\alpha F^u(Y_s)\,ds\right] \Rightarrow J^u(y) = E^y\left[\int_0^\alpha F^u(Y_s)\,ds\right] + E^y\left[J^u(Y_\alpha)\right]$$

$$\Rightarrow B(y) \geq E^y\left[\int_0^\alpha F^v(Y_s)\,ds\right] + E^y\left[B(Y_\alpha)\right]$$

$\forall v$

By Dynkin's formula:

$$E^y[B(Y_\alpha)] = B(y) + E^y\left[\int_0^\alpha (L^v B)(Y_s)\,ds\right] \Rightarrow$$

$$\Rightarrow \quad B(y) \geq E^y\left[\int_0^\alpha F^v(Y_s)\,ds\right] + B(y) + E^y\left[\int_0^\alpha (L^v B)(Y_s)\,ds\right.$$

$$\Rightarrow \quad E^y\left[\int_0^\alpha \left(F^v(Y_s) + (L^v B)(Y_s)\right)ds\right] \leq 0$$

We choose a proper α to get the teorem.

the idea is $\alpha \longrightarrow 0$

$$\Rightarrow \quad F^v(Y_s) + (L^v B)(Y_s) \leq 0$$

+ optimal control $\Rightarrow$ the eqs because.

This proof works as long as we choose a proper control α.